THE APPROXIMATE INSCRIPTION OF CERTAIN REGULAR POLYGONS.

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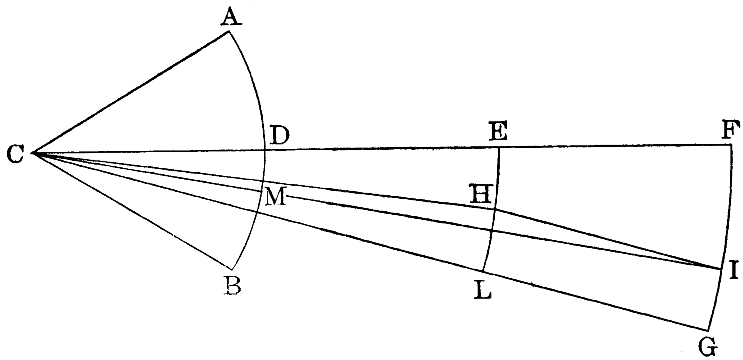
1. *To inscribe a regular nonagon in a circle.*—Let ACB be a central angle of 60° in the given circle. Bisect ACB by CF, laying off DE and EF equal to CD.



Bisect FCB by CG, and draw the arcs EL and FG from C as a centre. Bisect FCG, thus determining H, the mid-point of EL. From H draw HI parallel to CG, and from the point I where it cuts FG draw IC. Then will ICG be nearly equal to one third of FCG, or 5°. By solving the triangle HIC we find that $CIH = ICG = 4^\circ\ 59'\ 31''.37$. Hence $ACM = 40^\circ\ 0'\ 28''.63$, and a chord drawn from A to M will be very nearly equal to the side of a regular nonagon inscribed in the circle whose radius is AC : $9 \times 28''.63 = 257''.67$, which is $\frac{1}{5030}$ of the entire circumference. If one half the angle FCI were subtracted from an angle of 45°, an angle of $39^\circ\ 59'\ 45''.68$ would result, which is a closer approximation to 40°.

It is evident that there are many ways of getting angles more nearly equal to 5°; we give a few, each of which depends upon the trisection of a small angle, together with successive bisections of angles; thus:

$$5^\circ = \tfrac{15^\circ}{2} - \tfrac{1}{3}\cdot\tfrac{15^\circ}{2} \qquad (A)$$

$$= \tfrac{15^\circ}{4} + \tfrac{1}{3}\cdot\tfrac{15^\circ}{4} \qquad (B)$$

$$= \tfrac{15^\circ}{4} + \tfrac{15^\circ}{8} - \tfrac{1}{3}\cdot\tfrac{15^\circ}{8} \qquad (C)$$

$$= \tfrac{15^\circ}{4} + \tfrac{15^\circ}{16} + \tfrac{1}{3}\cdot\tfrac{15^\circ}{16} \qquad (D)$$

$$= \tfrac{15^\circ}{4} + \tfrac{15^\circ}{16} + \tfrac{15^\circ}{32} - \tfrac{1}{3}\cdot\tfrac{15^\circ}{32} \qquad (E)$$

$$= \tfrac{15^\circ}{4} + \tfrac{15^\circ}{16} + \tfrac{15^\circ}{64} + \tfrac{1}{3}\cdot\tfrac{15^\circ}{64}, \qquad (F)$$

. . . .

If the trisections included in solutions $(A), \dots (F)$ be performed in the same manner in which FCG was trisected (getting ICG as the approximate value of $\frac{1}{3}FCG$), it is easy to compute the error of each solution in the following way:—

Let FCG now denote any one of these angles to be trisected; denote ICG by x and HCG by y; let $2F(y)$ denote $y - \sin y$, and $2F(x)$ denote $x - \sin x$.

Since $\sin y = \frac{3}{2} \sin x$, we have

$$y - 2F(y) = \frac{3}{2}x - \frac{3}{2} . 2F(x). \tag{1}$$

By development,

$$2F(y) = \frac{1}{6}\sin^3 y + \frac{3}{40}\sin^5 y + \frac{5}{112}\sin^7 y + \frac{35}{1152}\sin^9 y + \dots . \tag{2}$$

Likewise, substituting $\frac{2}{3}\sin y$ for $\sin x$, we get

$$2F(x) = \frac{4}{81}\sin^3 y + \frac{4}{405}\sin^5 y + \frac{40}{15309}\sin^7 y + \frac{140}{177147}\sin^9 y + \dots . \tag{3}$$

Hence

$$\frac{2F(x)}{2F(y)} = \frac{8}{27} - \left(\frac{2}{27}\sin^2 y + \frac{1549}{51030}\sin^4 y + \frac{651827}{41334300}\sin^6 y + \dots\right) = \frac{8}{27} - \varphi(y), \tag{4}$$

where $\varphi(y)$ denotes the series in the ( ).

Find the value of $F(x)$ from (4), substitute it in (1), and reduce, obtaining

$$y - \frac{3}{2}x = \frac{5}{9} . 2F(y) + \frac{3}{2}\varphi(y) . 2F(y),$$

or

$$\frac{2}{3}y - x = \frac{10}{27} . 2F(y) + \varphi(y) . 2F(y). \tag{5}$$

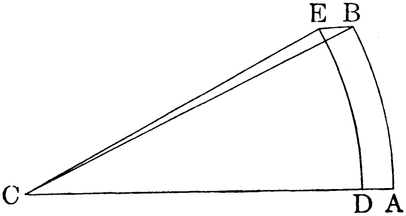
Now $\frac{2}{3}y - x$ is the error of x, and (5) furnishes an easy method of computing this error. Thus we find the following table:—

Solution.	$2y$	Error of x.	$\frac{9 \times \text{error of } x}{1,296,000''}$
	°	''	
(A)	7.5	3.573	1 : 40,290±
(B)	3.75	0.4462	1 : 322,700±
(C)	1.875	0.0558	1 : 2,583,000±
(D)	0.9375	0.00698	1 : 20,660,000±
(E)	0.46875	0.00087	1 : 165,000,000±
(F)	0.234375	0.00011	1 : 1,300,000,000±

It thus becomes apparent that it is possible to find a third of any acute angle without much labor, and attain a result the theoretical error of which is far less than the unavoidable errors of the most careful draughtsmen with the best drawing instruments manufactured. When the angle is between 45° and

90°, it is best to find one third of its complement, and subtract it from 30°. If the given angle is obtuse it is best to find a third of its supplement, and subtract it from 60°.

2. *The Inscription of a Regular Polygon of Eleven Sides.—1st Method.*—

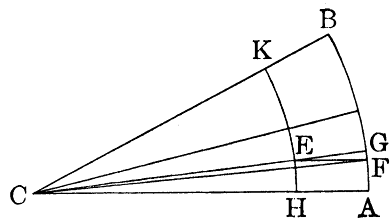


Divide the radius CA of the given circle into twelve equal parts, of which CD contains eleven. Construct $ACB = 30°$. Draw AB and DE. From B draw BE parallel to AC. Draw CE. Then $ECD = 33° \, 3' \, 20''.+$, which is one per cent. larger than $360° : 11$. If the arc ED had been drawn with a radius of 11.1, ECD would have been equal to $32° \, 43' \, 13''.60$, which is $\frac{1}{4793}$ smaller than $360° : 11$.

2d Method.—If a triangle be formed of sides 6, 10, 11, the angle opposite 6 will be $32° \, 45' \, 50''$, which is $\frac{1}{897}$ larger than $360° : 11$.

3. *The Inscription of a Regular Polygon of Thirteen Sides.*—In the preceding figure let $CD = 12$, and $CA = 13$, and the angle $ECD = 30°$, EB being parallel to AC. Then $ACB = 27° \, 29' \, 11''.14$, which is $\frac{10}{1345}$ smaller than $360° : 13$. If CA be taken as $12\frac{10}{11}$, ACB will be $15''.27$ larger than desired, or $\frac{1}{6529}$ too large.

4. *The Multisection of Small Angles.*—If we wish to find one fifth of ACB, by successive bisections we obtain $ACG = \frac{1}{4} BCA$. Lay off $CH = 4$ and $CA = 5$, and draw the arcs BA and HK. From E, where CG cuts HK, draw EF parallel to CA. FCA is nearly one fifth of BCA. Denote GCA by a, and FCA by x. By reasoning similar to that employed before we may show that



$$\tfrac{4}{5} a - x = \left(\tfrac{36}{125} + \tfrac{1296}{15625} \sin^2 a + \ldots\right) 2F(a).$$

If $BCA = 15°$, FCA is $2° \, 59' \, 57''.22$, which is $2''.78$ too small.

To find $\frac{1}{7}$ of BCA. By successive bisections, find $FCA = \frac{1}{8} BCA$. Lay off $CA = 8$ and $CH = 7$, and draw BA and KH; also draw FE parallel to CA. ECA is nearly equal to $\frac{1}{7} BCA$. Denoting FCA by a, and ECA by x, we find

$$\tfrac{8}{7} a - x = \left(-\tfrac{120}{343} - \tfrac{3456}{16807} \sin^2 a - \ldots\right) 2F(a).$$

If $BCA = 15°$, $ECA = 2° \, 8' \, 34''.71$, which is $0''.42$ too large.

By combinations of the methods which have been set forth one may easily find $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, etc. of any small angle with considerable accuracy. These methods are so simple that it is quite probable that they are not new; but I have never heard of them.